Section 4.7

Limits of Indeterminate Form

Indeterminate forms appear when we attempt to apply algebraic limit laws but we get an algebraic expression that can not be computed. The first of these come from quotients:

Definition 4.7.1. Let f and g be functions defined near a. If

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0,$$

then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of indeterminate form $\dfrac{0}{0}$. (Think $\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{0}{0}$)

If

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty,$$

then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of indeterminate form $\dfrac{\infty}{\infty}$. (Think $\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{\infty}{\infty}$)

Remember $\dfrac{a}{0}$ is not indeterminate!

Example 4.7.2. A limit we can already compute that is indeterminate: $\lim_{x\to 2} \dfrac{x^2 - x - 2}{x - 2}$

Theorem 4.7.3 (L'Hôpital's Rule)**.** Let f and g be differentiable on an open interval I containing a with $g(x) \neq 0$ on I. If $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is of indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists or is $\pm\infty$. The rule also holds for $x \to a^{\pm}$ and $x \to \pm\infty$.

Sketch.

Example 4.7.4.

1. $\lim_{x \to 0} \dfrac{3 \sin 4x}{5x}$

2. $\lim_{x \to \infty} \dfrac{e^{\frac{1}{x}} - 1}{\frac{1}{x}}$

3. $\lim_{x \to 1} \dfrac{x^2 + 2x}{x + 3}$

4. $\lim_{x\to\infty} \dfrac{e^{3x}}{3e^{3x}+5}$

Other Indeterminate Forms

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then $\lim_{x\to a} f(x)g(x)$ is indeterminate of the form $0 \cdot \infty$.

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x) - g(x)$ is indeterminate of the form $\infty - \infty$.

Example 4.7.5.

1. $\lim_{x\to\infty} x^2 \sin\left(\dfrac{1}{4x^2}\right)$

2. $\lim_{x \to 0^+} \left(\cot x - \frac{1}{x} \right)$

For $\lim_{x \to a} f(x)^{g(x)}$, there are three indeterminate forms: $1^\infty, 0^0$, and ∞^0.

Example 4.7.6.

1. $\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^x$

2. $\lim_{x \to \infty} x^{\frac{1}{x}}$

3. $\lim_{\theta \to 0^+} (\sin \theta)^{\tan \theta}$

Indeterminate Forms: $\frac{0}{0}\,,\frac{\infty}{\infty}\,,0\cdot\infty\,,\infty-\infty\,,1^{\infty}\,,0^{0}\,,\infty^{0}$

Not Indeterminate: $\frac{a}{0}\to\pm\infty\,,\ \frac{a}{\infty}\to 0\,,\ \infty\cdot\infty\to\infty\,,\ 1^{1}=1,1^{0}=1,0^{1}=0,$ $\infty^{\infty}\to\infty$